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A CLASSIFICATION OF COVARIANTS
AND CONTRAVARIANTS OF PLANE CUBICS

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CHAPTER 1

Introduction

Let the group SL_3 act on V_3 , a three-dimensional vector space over the field, k , (k is alg.closed, $\text{char } k = 0$). This induces an action of SL_3 on $\text{Sym } V_3$, and also on $\text{Sym}(\text{Sym}_3 V_3)$, and hence on $\text{Sym}(\text{Sym}_3 V_3) \otimes_k \text{Sym } V_3$, which is isomorphic to a bigraded polynomial ring.

Similarly we obtain an action on $\text{Sym}(\text{Sym}_3 V_3) \otimes_k \text{Sym}(\text{Sym}_1 V_3)$, which is also isomorphic to a bigraded polynomial ring.

We will find a set of ring generators over k , and the relations between them, for the subring of $\text{Sym}(\text{Sym}_3 V_3) \otimes_k \text{Sym } V_3$, which is invariant with respect to the action of SL_3 . In classical notation (i.e. Salmon), the bihomogeneous elements of this subring are called covariants. I will also describe the corresponding invariant subring of $\text{Sym}(\text{Sym}_3 V_3) \otimes \text{Sym}(\text{Sym}_1 V_3)$. The bihomogeneous elements of this ring are called contravariants in classical notation. A subring of both these invariant subrings is the ring of invariant elements of $\text{Sym}(\text{Sym } V_3)$. The homogeneous elements of this ring are called invariants in classical notation. The classification of generators of this ring is a corollary of both the preceeding more general classifications. An easier proof, which I here omit, is also possible.

All these problems are special cases of classifying the invariant elements of the trigraded ring

$$\text{Sym}(\text{Sym}_3 V_3)_k \otimes \text{Sym}(\text{Sym}_1 V_3)_k \otimes \text{Sym} V_3.$$

The trihomogeneous invariant elements of this ring are called mixed concomitants in classical notation. This general problem is not solved here.

The results that I will show in this paper, are partly suggested by Salmon, see [1], but he has not given any exact proofs in this book.

The last chapter in this paper is devoted to showing the connection between the invariants described here, and the so-called j-invariant, described f.ex. by Hartshorne, see [3].

CHAPTER 2

MAIN THEOREMS.

2.1 A description of the action of GL_3 .

$$\text{Let } g \in GL_3 \text{ be given by } \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \quad V = (v_1, v_2, v_3) \in V_3.$$

We define:

$$g \cdot v = (\lambda_{11}v_1 + \lambda_{12}v_2 + \lambda_{13}v_3, \lambda_{21}v_1 + \lambda_{22}v_2 + \lambda_{23}v_3, \lambda_{31}v_1 + \lambda_{32}v_2 + \lambda_{33}v_3).$$

When $p \in \text{Sym} V_3$, which is isomorphic to $k[x, y, z]$,

define $g \cdot p = p(g^{-1}v)$.

This gives:

$$x = \lambda_{11}g \cdot x + \lambda_{12}g \cdot y + \lambda_{13}g \cdot z$$

$$y = \lambda_{21}g \cdot x + \lambda_{22}g \cdot y + \lambda_{23}g \cdot z$$

$$z = \lambda_{31}g \cdot x + \lambda_{32}g \cdot y + \lambda_{33}g \cdot z.$$

Let α, β, γ be the generators of $\text{Sym}(\text{Sym}_1 V_3)$ over k . Similarly we obtain:

$$g \cdot \alpha = \lambda_{11} \alpha + \lambda_{21} \beta + \lambda_{31} \gamma$$

$$g \cdot \beta = \lambda_{12} \alpha + \lambda_{22} \beta + \lambda_{32} \gamma$$

$$g \cdot \gamma = \lambda_{13} \alpha + \lambda_{23} \beta + \lambda_{33} \gamma ,$$

where "the general line" is given by $\alpha x + \beta y + \gamma z = 0$.

Correspondingly we let "the general cubic in \mathbb{P}_k^2 " be given by $ax^3 + by^3 + cz^3 + 3a_2 x^2 y + 3a_3 x^2 z + 3b_1 xy^2 + 3b_2 y^2 z + 3c_1 xz^2 + 3c_2 yz^2 + 6mxyz$.

We interpret a, \dots, m as the generators of $\text{Sym}(\text{Sym}_3 V_3)$. Similar to the above obtained actions of GL_3 , we get:

$$g \cdot a = a \lambda_{11}^3 + b \lambda_{21}^3 + c \lambda_{31}^3 + 3a_2 \lambda_{11}^2 \cdot \lambda_{21} + 3a_3 \lambda_{11}^2 \cdot \lambda_{31} + \dots + 6m \lambda_{11} \lambda_{21} \lambda_{31} ,$$

and so on for $g \cdot b, \dots, g \cdot m$. Common to all 10 is that the degree in the set $\lambda_{11}, \dots, \lambda_{33}$ is 3.

We now define:

$$R \stackrel{\text{def}}{=} k[a, \dots, m, x, y, z] \approx \text{Sym}(\text{Sym}_3 V_3) \otimes_k \text{Sym} V_3$$

$$V \stackrel{\text{def}}{=} k[a, \dots, m, \alpha, \beta, \gamma] \approx \text{Sym}(\text{Sym}_3 V_3) \otimes_k \text{Sym}(\text{Sym}_1 V_3)$$

$$W \stackrel{\text{def}}{=} k[a, \dots, m] \approx \text{Sym}(\text{Sym}_3 V_3).$$

We define the bidegree of a bihomogeneous element of R to be (G, o) , if the degree in a, \dots, m is G , and the degree (order) in x, y, z is o .

We define the bidegree of a bihomogeneous element of V to be (G, u) , if the degree in a, \dots, m is G , and the degree (order) in α, β, γ is u .

For homogeneous elements of W we simply have a degree, G .

2.2. Theorems

We will prove these theorems:

Theorem 2.2.1.

$R^{SL_3} = k[S, T, U, H, \Theta, J]$, where S, T, U, H, Θ, J have bidegrees $(4,0), (6,0), (1,3), (3,3), (8,6), (12,9)$ resp. The only algebraic relation between these is:

$$J^2 = 4\Theta^3 + T^2U^2\Theta^2 + \Theta(-4S^3U^4 + 2STU^3H - 72S^2U^2H^2 - 18TUH^4 + 108SH^4) \\ - 16S^4U^5H - 11S^2TU^4H^2 - 4T^2U^3H^3 + 54STU^2H^4 - 432S^2UH^5 - 27TH^6 .$$

A more precise description of S, T, H, U, Θ, J follows in § 2.4.

Theorem 2.2.2.

$V^{SL_3} = k[S, T, P, Q, F, K]$, where S, T, P, Q, F, K have bidegrees: $(4,0), (6,0), (3,3), (5,3), (4,6), (12,9)$ resp. The only algebraic relation between these is:

$$K^2 = \frac{1}{16}(T^2 + 64S^3)F^3 - 1/8 TF^2Q^2 - 12S^2F^2PQ + \frac{9}{2}STF^2P^2 + \frac{1}{16} FQ^4 \\ + \frac{15}{2} SFQ^2P^2 - \frac{9}{2} TFQP^3 - 27S^2FP^4 + \frac{1}{2} Q^3P^3 + 54SQP^5 - 27TP^6 .$$

A closer description of P, Q, F, K follows in § 2.4.

Corollary 2.2.3.

$W^{SL_3} = k[S, T]$. No algebraic relations between S and T .

2.3. A restatement of the property of being a co/contravariant.

We recall that the bihomogeneous elements of R^{SL_3} and V^{SL_3} were called covariants and contravariants respectively.

It can be easily shown that for a bihomogeneous polynomial $C \in R$, the property of being a covariant is equivalent to:

$$(2.3.1) \quad C(g \cdot a, \dots, g \cdot m, g \cdot x, g \cdot y, g \cdot z) = (\det g)^w C(a, \dots, m, x, y, z)$$

for all $g \in \text{GL}_3$, and $3w = 3G - o$, where bidegree of C is (G, o) .

Similarly for a bihomogeneous $B \in V$, bidegree $B = (G, u)$, the property of being a contravariant can be expressed as:

$$(2.3.2) \quad B(g \cdot a, \dots, g \cdot m, g \cdot \alpha, g \cdot \beta, g \cdot \gamma) = (\det g)^w B(a, \dots, \gamma)$$

for all $g \in \text{GL}_3$, and $3w = 3G + u$. It also follows easily that $w \in \mathbb{N}_0$. w is called the weight in both cases.

We now take (2.3.1) and (2.3.2) as definitions of covariants and contravariants respectively.

2.4. A description of $S, T, U, H, \Theta, J, P, Q, F$ and K .

$$U \stackrel{\text{def}}{=} ax^3 + by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_1xy^2 + 3b_3y^2z + 3c_1xz^2 + 3c_2yz^2 + 6mxyz.$$

We refer to U as "the curve itself" or as "the general cubic".

$H \stackrel{\text{def}}{=}$ the Hessian of U . The formula for H is given in [1], page 183.

It can be shown that the zeroes of a covariant in \mathbb{P}_k^2 is a set of points, related to the "original curve", U , s.t. the zeroes do not change, even if we change the systems of coordinates (act with a $g \in \text{GL}_3$). Conversely will any coordinate-free geometrical algebraic property be described by the locus of one or more covariants.

Such a geometrical property (for details, see [1] and [2] gives rise to another covariant, Θ , of bidegree (8,6). I don't know the general formula for this, but I will give the so-called canonical form of Θ in 2.5.

In each of the 9 inflexional points of a non-singular cubic (U), the polar conic splits up into the inflexional tangent and another line, the so-called harmonic polar. The product of these nine harmonic polars is a covariant of bidegree (12,9). We call this J. See 2.5.

In [2], Salmon defines a so-called symbolic method from which all invariants and covariant can be defined. A proof of this is given in [4]. Symbolically we define

$$S \stackrel{\text{def}}{=} \overline{123} \overline{234} \overline{341} \overline{412}, \text{ and}$$

$$T \stackrel{\text{def}}{=} \overline{123} \overline{124} \overline{235} \overline{316} \overline{456}^2. \text{ The degrees are 4 and 6 resp.}$$

For fomulas, see [1].

We now define the contravariants K, P, Q and F. F is defined as the dual curve of U (bidegree (4,6)). K is defined as the product of the 9 inflexional points of U, considered as lines in the dual space (bidegree (12,9)). P is defined as the first evectant of S. (bidegree (3,3)). Q is defined as the first evectant of T. (bidegree (5,3)). For definitions of evectants, see [2].

2.5. Description of canonical form.

Choosing a proper system of coordinates, a nonsingular cubic can be written:

$$U = x^3 + y^3 + z^3 + 6mxyz.$$

There are in general 12 possible m 's for a given curve. Any other covariant/contravariant/invariant can also be written on such a canonical form, since they are defined by a, \dots, m defining U . On canonical form, we obtain:

$$S_{\text{can}} = m - m^4$$

$$T_{\text{can}} = 1 - 20m^3 - 8m^6$$

$$H_{\text{can}} = -m^2(x^3 + y^3 + z^3) + (1 + 2m^3)xyz$$

$$J_{\text{can}} = (1 + 8m^3)^3(z^3 - x^3)(y^3 - z^3)(x^3 - y^3)$$

$$\begin{aligned} \Theta_{\text{can}} &= 3m^3(1 + 2m^3)(x^3 + y^3 + z^3)^2 - m(1 - 20m^3 - 8m^6)xyz(x^3 + y^3 + z^3) \\ &\quad - 3m^2(1 - 20m^3 - 8m^6)(xyz)^2 - (1 + 8m^3)^2(y^3x^3 + y^3z^3 + x^3z^3) \end{aligned}$$

$$P_{\text{can}} = m(\alpha^3 + \beta^3 + \gamma^3) + (1 - 4m^3)\alpha\beta\gamma.$$

$$Q_{\text{can}} = (1 - 10m^3)(\alpha^3 + \beta^3 + \gamma^3) - (30m^2 + 24m^5)\alpha\beta\gamma$$

$$K_{\text{can}} = (1 + 8m^3)^3(\gamma^3 - \alpha^3)(\beta^3 - \gamma^3)(\alpha^3 - \beta^3)$$

$$\begin{aligned} F_{\text{can}} &= \alpha^6 + \beta^6 + \gamma^6 - (2 + 32m^3)(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3) - (24m + 48m^4)(\alpha\beta\gamma)^2 \\ &\quad - 24m^2\alpha\beta\gamma(\alpha^3 + \beta^3 + \gamma^3). \end{aligned}$$

CHAPTER 3

PROOF OF THEOREMS, 1ST PART

In this chapter we will find sets of generators for the rings R^{SL_3} and V^{SL_3} .

3.1. Some technical lemmas.

For a co/contravariant C , we denote its canonical form by C_{can} .

Lemma 3.1.1.

If A and B are co/contravariants, and A_{can} divides B_{can} and A neither possesses multiple factors nor has the discriminant of U, D , as factor, then A divides B in general. ($D = T^2 + 64S^3$).

Proof of lemma 3.1.1.

It is enough to prove $A = 0 \Rightarrow B = 0$. Pick a set of values a, \dots, m , and an associated "canonical" value for m, m_0 . This is possible if a, \dots, m represents a canonical curve. We assume for simplicity of notation that A is a covariant.

$$\begin{aligned} A(a, \dots, m, x, y, z) = 0 &\Rightarrow A_{\text{can}}(m_0, x, y, z) = 0 \\ \Rightarrow B_{\text{can}}(m_0, x, y, z) = 0 &\Rightarrow B(a, \dots, m, x, y, z) = 0 \end{aligned}$$

Look at the hypersurface $A = 0$ in \mathbb{A}_k^{13} (or $\mathbb{P}_k^9 \times \mathbb{P}_k^2$)

$B|_{A=0} = 0$ on a dense open set on $A = 0$. (i.e. on each of its irreducible components), since $D \cap A$ has $\text{codim} \geq 2$. Therefore:
 $B = 0$ on $A = 0$. Q.E.D.

Lemma 3.1.2.

a.) $m, U_{\text{can}}, H_{\text{can}}, @_{\text{can}}$ are alg.indep./ k .

b.) $m, P_{\text{can}}, Q_{\text{can}}, F_{\text{can}}$ are alg.indep./ k .

Proof: a.) The elementary symmetric functions in x^3, y^3, z^3 ; $x^3 + y^3 + z^3, x^3y^3 + x^3z^3 + y^3z^3, (xyz)^3$ can be written as rational functions in $m, U_{\text{can}}, H_{\text{can}}$ and $@_{\text{can}}$.

Similarly for b.)

Q.E.D.

Lemma 3.1.3.

For a covariant C , bidegree $(G,0)$, weight $= w$, we have,

$$a.) \quad (1+2m)^G C_{\text{can}}\left(\frac{1-m}{1+2m}, g_1 x, g_1 y, g_1 z\right) = (\epsilon^2 - \epsilon)^w C_{\text{can}}(m, x, u, z),$$

$$\text{where } \epsilon = e^{i \cdot 2\pi/3}, \quad g_1 = 3^{-1/3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon \end{pmatrix}.$$

$$b.) \quad (1+2\epsilon m)^G C_{\text{can}}\left(\frac{1-\epsilon m}{1+2\epsilon m}, g_2 x, g_2 y, g_2 z\right) = (\epsilon^2 - \epsilon)^w C_{\text{can}}(m, x, y, z),$$

$$\text{where } g_2 = 3^{-1/3} \begin{pmatrix} 1 & 1 & 1 \\ \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & 1 & \epsilon \end{pmatrix}.$$

$$c.) \quad (1+2\epsilon^2 m)^G C_{\text{can}}\left(\frac{1-\epsilon^2 m}{1+2\epsilon^2 m}, g_3 x, g_3 y, g_3 z\right) = (\epsilon^2 - \epsilon)^w C_{\text{can}}(m, x, y, z),$$

$$\text{where } g_3 = 3^{-1/3} \begin{pmatrix} 1 & 1 & 1 \\ \epsilon & \epsilon^2 & 1 \\ \epsilon & 1 & \epsilon^2 \end{pmatrix}.$$

d.) Exchange x, y, z with α, β, γ resp. in a.), b.), and c.).

Then a., b., and c. applies to contravariants, C , as well.

Proof of lemma 3.1.3:

A direct calculation using formulas (2.3.1) and (2.3.2).

We now introduce the concepts x -weight, y -weight, z -weight, where each of these are defined for $a, \dots, m, x, y, z, \alpha, \beta, \gamma$. The x -weight of a product is the sum of x -weights of its factors. Similarly for y -weights and z -weights.

We have the following table (definition):

	a	b	c	a ₂	a ₃	b ₁	b ₃	c ₁	c ₂	m	x	y	z	α	β	γ
x-w.	3	0	0	2	2	1	0	1	0	1	-1	0	0	1	0	0
y-w.	0	3	0	1	0	2	2	0	1	1	0	-1	0	0	1	0
z-w.	0	0	3	0	1	0	1	2	2	1	0	0	-1	0	0	1

Lemma: 3.1.4:

Let C be a co/contravariant of weight w . For all terms of C we have: $x\text{-weight} = y\text{-weight} = z\text{-weight} = w$.

Proof of lemma 3.1.4.

Act on C with $g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, λ arbitrary in k , and use formulas (2.3.1) and (2.3.2).

Similarly when $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$.

3.2. Gradual approach to finding sets of generators for R^{SL_3} and V^{SL_3} .

We now list a sequence of lemmas that will bring us nearer to the conclusions of theorems 2.2.1 and 2.2.2.

Lemma 3.2.1.

a.) For a covariant, C , we have:

$$C_{can} = P_0(m, x^3, y^3, z^3) + xyz \cdot P_1(m, x^3, y^3, z^3) + (xyz)^2 P_2(m, x^3, y^3, z^3),$$

where P_0, P_1, P_2 are polynomials in 4 indeterminates.

b.) For a contravariant, B , we have:

$$B_{\text{can}} = P_0(m, \alpha^3, \beta^3, \gamma^3) + (\alpha\beta\gamma)P_1(m, \alpha^3, \beta^3, \gamma^3) + (\alpha\beta\gamma)^2P_2(m, \alpha^3, \beta^3, \gamma^3),$$

where P_0, P_1, P_2 are polynomials in 4 indeterminates.

Lemma 3.2.2.

a.) For covariants, C , where weight C is even, we have:

$$C_{\text{can}} = \mathcal{P}(m, x^3 + y^3 + z^3, xyz, x^3y^3 + x^3z^3 + y^3z^3),$$

where \mathcal{P} is a polynomial in 4 indeterminates.

b.) For contravariants, B , where weight B is even, we have:

$$B_{\text{can}} = \mathcal{P}(m, \alpha^3 + \beta^3 + \gamma^3, \alpha\beta\gamma, \alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3),$$

where \mathcal{P} is a polynomial in 4 indeterminates.

Lemma 3.2.3.

a.) For even-weighted covariants, C , we have:

$$C_{\text{can}} = \sum_{\text{finite}} m^l U_{\text{can}}^i \cdot H_{\text{can}}^j \cdot \Theta_{\text{can}}^k \cdot q_{ijkl} \quad (3.2.1.)$$

b.) For even-weighted contravariants, B , we have:

$$B_{\text{can}} = \sum_{\text{finite}} m^l P_{\text{can}}^i \cdot Q_{\text{can}}^j \cdot F_{\text{can}}^k \cdot c_{ijkl} \quad (3.2.2.)$$

Lemma 3.2.4.

a.) If an even-weighted covariant, C , is written in the form (3.2.1.), and there exist terms with $l = 0$, we have for the corresponding i, j, k -values:

$$G-i-3j-8k \equiv 0 \pmod{6}, \text{ and } G-i-3j-8k \geq 0, \text{ where bidegree } C = (G, 0).$$

b.) If an even-weighted contravariant, B , is written in the form (3.2.2.), and there exist terms with $l = 0$, we have for the corresponding i, j, k -values:

$$G - 3i - 5j - 4k \equiv 0 \pmod{6}, \text{ and } G - 3i - 4k - 5j \geq 0, \text{ where bidegree } B = (G, u).$$

Lemma 3.2.5.

a.) Assume C is an even-weighted covariant. Referring to formula (3.2.1.), we have:

$C = \sum_{l=0} q_{ijkl} T^{\frac{G-i-3j-8k}{6}} \cdot U^i H^j \otimes^k$ is a covariant that has S among its factors.

b.) Assume B is an even-weighted contravariant. Referring to formula (3.2.2.) we have:

$B = \sum c_{ijkl} T^{\frac{G-3i-5j-4k}{6}} P^i Q^j F^k$ is a contravariant that has S among its factors.

Proposition 3.2.6.

a.) All even-weighted covariants, C , can be written on the form:

$$C = \sum q_{ijklm} S^i T^j U^k \otimes^m H^l, \text{ where}$$

$$4i + 6j + k + 3l + 8m = G, \quad 3k + 3l + 6m = 0, \text{ and bidegree } C = (G, 0).$$

b.) All even-weighted contravariants, B , can be written on the form:

$$B = \sum c_{ijklm} S^i T^j P^k Q^l F^m, \text{ where}$$

$$4i + 6j + 3k + 5l + 4m = G, \quad 3k + 3l + 6m = u, \text{ and bidegree } B = (G, u).$$

Lemma 3.2.7.

- a.) An odd-weighted covariant has J among its factors.
 b.) An odd-weighted contravariant has K among its factors.

The quotient then is an even-weighted co- or contravariant resp., and we have found sets of generators for R^{SL_3} and V^{SL_3} , provided the sequence of lemmas are proved.

3.3. Proof of the sequence of lemmas listed in section 3.2.

Proof of lemma 3.2.1.

$$\text{Put } C_{\text{can}} = \sum q_{ijkl} m^l x^i y^j z^k. \quad (3.3.1)$$

$$\epsilon = e^{i \cdot 2\pi/3}.$$

$$\text{We let } g_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \quad \text{act}$$

on C . Applying (2.3.1), we get:

$$(3.3.2): \quad 2i+1 \equiv 2j+1 \equiv 2k+1 \equiv w \pmod{3} \quad \text{for all terms in (3.3.1).}$$

This implies $i \equiv j \equiv k \pmod{3}$, which gives us the conclusion of lemma 3.2.1 a. For contravariants the analogous formula to (3.3.2) is $i+1 \equiv j+1 \equiv k+1 \equiv \quad \pmod{3}$ which gives the conclusion of lemma 3.2.1. b.

Proof of lemma 3.2.2:

An even-weighted co- or contravariant is symmetric in x, y and z over $k[m]$ on canonical form. This makes P_0, P_1 and P_2 of lemma 3.2.1 to be symmetric in their 3 last arguments (indeterminates). This gives the conclusion of lemma 3.2.2.

Proof of lemma 3.2.3.

This part of the sequence is the hardest to prove. In lemma 3.2.1 we used the fact that we can "stretch" each axis with a factor ϵ , and still maintain the canonical form. In lemma 3.2.2 we used the fact that we could permute the axis. Here we will change to another system of lines connecting the 9 inflexional points. There are 4 such systems altogether.

Lemma 3.2.2 enables us to write a co- or contravariant (C or B) on the canonical form like this:

$$C_{\text{can}} = \sum_{\text{finite}} m^l (x^3 + y^3 + z^3)^i (xyz)^j (x^3 y^3 + x^3 z^3 + y^3 z^3)^k \cdot q_{ijkl} \quad (3.3.3)$$

$$B_{\text{can}} = \sum_{\text{finite}} m^l (\alpha^3 + \beta^3 + \gamma^3)^i (\alpha\beta\gamma)^j (\alpha^3 \beta^3 + \alpha^3 \gamma^3 + \beta^3 \gamma^3)^k \cdot C_{ijkl}.$$

This gives:

$$C_{\text{can}} = \sum_{ijkl} q_{ijkl} \cdot \frac{m^l [(1+2m^3)U_{\text{can}} - 6mH_{\text{can}}]^i [m^2 U_{\text{can}} + H_{\text{can}}]^j}{(1+8m^3)^{i+j+2k}}.$$

$$[m^3(2+m^3)U_{\text{can}}^2 - m(1+2m^3)U_{\text{can}} \cdot H_{\text{can}} - 3m^2 H_{\text{can}}^2 - \Theta_{\text{can}}]^k, \text{ and}$$

$$B_{\text{can}} = \sum_{ijkl} C_{ijkl} \frac{m^l [(30m^2 + 24m^5)P_{\text{can}} + (1-4m^3)Q_{\text{can}}]^i [(1-10m^3)P_{\text{can}} - mQ_{\text{can}}]^j}{(1+8m^3)^{2i+2j+4k}}.$$

$$[(48m^6 + 207m^3 - 6)mP_{\text{can}}^2 + m^2(21-48m^3)P_{\text{can}}Q_{\text{can}} + \frac{1}{4}(1-16m^3)Q_{\text{can}}^2 - \frac{1}{4}(1+8m^3)^3 F_{\text{can}}]^k.$$

We now multiply the two last equations with $D_{\text{can}}^r = (1+8m^3)^{3r}$, where D is the discriminant, and r is a large integer (large enough to make the denominators vanish). This gives:

$$(D^r C)_{\text{can}} = p(m, U_{\text{can}}, H_{\text{can}}, \Theta_{\text{can}}), (D^r B)_{\text{can}} = q(m, P_{\text{can}}, Q_{\text{can}}, F_{\text{can}}) \quad (3.3.4)$$

for suitable polynomials p and q .

We now use lemma 3.2.2 on the covariants $D^r C, U, H, \Theta$ and on the contravariants, $D^r B, P, Q, F$ to show that we can "cancel" the factor $[(1+2m)(1+2\epsilon m)(1+2\epsilon^2 m)]^r$ on the left sides of (3.3.4.)

Lemma 3.2.2.a and calculation gives:

$$\begin{aligned} (\epsilon^2 - \epsilon)^{W+12r} (D^r C)_{\text{can}} &= \\ &= \sum_{ijkl} q_{ijkl} (1+2m)^{G+12r} \cdot \left[\frac{(1+2m)^3 + 2(1-m)^3}{(1+2m)^4} U_{\text{can}} + 18 \frac{1-m}{(1+2m)^4} H_{\text{can}} \right]^i \cdot \\ &\cdot \left[\frac{(1-m)^2}{(1+2m)^3} U_{\text{can}} - \frac{3}{(1+2m)^3} H_{\text{can}} \right]^j \cdot \frac{(1-m)^1}{(1+2m)^1} \cdot \left[\frac{(1+2m)^3 + 8(1-m)^3}{(1+2m)^3} \right]^{3r-0/3} \cdot \\ &\left[\frac{(1-m)^3 (2(1+2m)^3 + (1-m)^3)}{(1+2m)^8} U_{\text{can}}^2 + \frac{3((1-m)(1+2m)^3 + 2(1-m)^4)}{(1+2m)^8} (UH)_{\text{can}} - \right. \\ &\left. - \frac{27(1-m)^2}{(1+2m)^8} H_{\text{can}}^2 + \frac{27}{(1+2m)^8} \Theta_{\text{can}} \right]^k. \end{aligned}$$

Bidegree $C = (G, 0)$.

This gives us $(1+2m)$ as factor $3r$ times on the left side, and $(G-1) - (i+2k) + 3r$ times on the right side for each combination i, j, k, l .

Corresponding usage of lemma 3.2.2.d. gives $3r$ times on the left side, and $3r + (G-1) - (2i+k)$ times on the right side for the contravariant, B .

Analogous usage of lemma 3.2.2, b, c, and d gives the same result for the factors $(1+2\epsilon m)$, and $(1+2\epsilon^2 m)$. Since $m, U_{can}, H_{can}, \Theta_{can}$ are alg.indep./k, and $m, P_{can}, F_{can}, Q_{can}$ are alg.indep./k, $k[m, U_{can}, H_{can}, \Theta_{can}]$ and $k[P_{can}, Q_{can}, F_{can}]$ are UFD's. This enables us to "cancel" factors successively in both rings, and the problem of showing the lemma is reduced to show:

$$(G-1) - (i+2k) \geq 0, \quad \text{for each term of } C_{can}$$

$$(G-1) - (2i+k) \geq 0, \quad \text{for each term of } B_{can},$$

corresponding to the expressions (3.3.3.).

We start with the covariant case, and repeat first part of

$$(3.3.3): C_{can} = \sum_{ijkl} q_{ijkl} \cdot m^l (x^3 + y^3 + z^3)^i (xyz)^j (x^3 y^3 + x^3 z^3 + y^3 z^3)^k,$$

which is a sum of products. One typical product contains the term:

$$q_{ijkl} \cdot m^l x^{3i+3k+j} y^{3k+j} z^j.$$

This monomial may be cancelled by another, coming from another product (but not from the same). For the moment, assume it is not being cancelled by any other monomial. Then it is a "canonical specialisation" of a term

$$a^s b^t c^v \cdot m^l x^{3i+3k+j} y^{3k+j} z^j \cdot q_{ijkl}.$$

We have: $G-1 = s+t+v$.

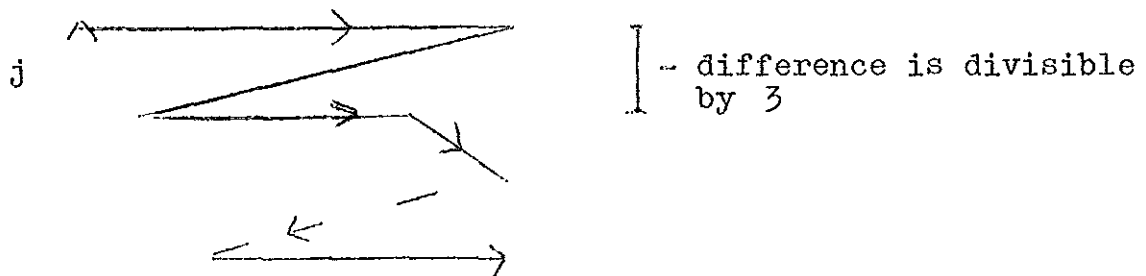
Lemma 3.1.4 gives: $t = v+k$, $s = v+k+i$, which gives:

$$G-1 = s+t+v = 3v+i+2k \geq i+2k.$$

Now we treat the "contravariant" case. In the same way we obtain a term like $a^s b^t c^v a^{3i+3k+j} b^{3k+j} c^j$. Lemma 3.1.4 gives:

$$s+i = t, \quad t+k = v, \quad \text{and} \quad G-1 = s+t+v \geq k+2i.$$

The proof is not finished yet, because the monomials we have referred to, may be cancelled by other monomials coming from other products corresponding to other i, j, k -combinations with the same l . The point is, however, that these new products will contain other monomials which are not cancelled by anyone from the "old" product. Also the "new" $(i+2k)$ - and $(2i+k)$ -value do not decrease, and this process of cancelling monomials and "substituting" them with new ones can not go on infinitely long. The task of verifying the existence of such a process is tedious, and I omit describing it in details here. It can be illustrated like this:



We just remark that the last, uncanceled term we end up with;

$$\pm x^{3i_1+3k_1+j_1} y^{3k_1+j_1} z^{j_1} \cdot m^1 \cdot q_{ijkl}$$

coming from a product;

$$\pm q_{ijkl} m^1 (x^3+y^3+z^3)^{i_1} (xyz)^{j_1} (x^3y^3+x^3z^3+y^3z^3)^{k_1},$$

has the property that $j_1 - j \leq 0$, $j_1 - j \equiv 0 \pmod{3}$,

$i_1 + j_1 + 2k_1 = i + j + 2k$. This applies also if we use α, β, γ instead of x, y, z respectively.

We now obtain: $G - 1 \geq 2k_1 + i_1 \geq 2k + i$

for covariants, and

$$G - 1 \geq 2i_1 + k_1 \geq 2i + k$$

for contravariants.

Q.E.D. lemma 3.2.3.

Proof of lemma 3.2.4.

Lemma 3.2.3 now gives:

$$C_{\text{can}} = \sum_{\substack{\text{finite} \\ i,j,k,l}} m^l U_{\text{can}}^i H_{\text{can}}^j \otimes_{\text{can}}^k q_{ijkl}, q_{ijkl} \in k \quad (3.3.4)$$

We were to prove that if there were any terms with $l = 0$, then for the corresponding i, j, k -values, $G - i - 3j - 8k \geq 0$, $G - i - 3j - 8k \equiv 0 \pmod{6}$. Now

$$C_{\text{can}}(0, x, y, z) = \sum_{l=0} (x^3 + y^3 + z^3)^i (xyz)^j (x^3 y^3 + x^3 z^3 + y^3 z^3)^k q_{ijk0}$$

The i, j, k 's in this summation are referring simultaneously to those in expression (3.3.3) and (3.3.4) with $l = 0$. The remarks at the end of the proof of lemma 3.2.3 ensure the existence of a term of the form

$$a^s b^t c^v x^{3i_1 + 3k_1 + j_1} y^{3k_1 + j_1} z^{j_1},$$

where $G = s + t + v = 3v + i_1 + 2k_1$, corresponding to each i, j, k -combination. This gives:

$$G - i - 3j - 8k = 3v + i_1 + 2k_1 - i - 3j - 8k = 3(v - j_1) + 2(j_1 - j) - 6k.$$

$(j_1 - j) \equiv 0 \pmod{3}$, so we have to prove that $3(v - j) \equiv 0 \pmod{2}$.

But: weight = $W = G - o/3 = 3v + i_1 + 2k_1 - i_1 - j_1 - 2k_1 = 3(u - j_1) + 2j_1$ is even, so $3(u - j_1)$ is even, and hence: $G - i - 3j - 8k \equiv 0 \pmod{6}$.

For contravariants lemma 3.2.3 gives:

$$B_{\text{can}} = \sum_{ijkl} m^l P_{\text{can}}^i \cdot Q_{\text{can}}^j \cdot F_{\text{can}}^k \cdot C_{ijkl}. \quad (3.3.5)$$

We were to prove that for those terms with $l = 0$, for the corresponding i, j, k -values:

$$G - 3i - 5j - 4k \geq 0, \quad G - 3i - 5j - 4k \equiv 0 \pmod{6}.$$

We look at:

$$B_{\text{can}}(0, \alpha, \beta, \gamma) = \sum_{\substack{ijk \\ n \leq k}} (\alpha\beta\gamma)^i (\alpha^3 + \beta^3 + \gamma^3)^{j+2n} (\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3)^{k-n} \cdot C_{ijkn}^1,$$

and a slightly modified version of the analogous proof for contravariants gives:

$$G - 3i - 5j - 4k \equiv 0 \pmod{6} \quad \text{for the current } i, j, k\text{-combinations.}$$

Application of lemma 3.2.2 gives that

$$C_{\text{can}} = \sum_{ijkl} (1+2m)^{G-i-3j-8k-l} U_{\text{can}}^i H_{\text{can}}^j \Theta_{\text{can}}^k \cdot (1-m)^l \cdot q_{ijkl}^1$$

for suitable $q_{ijkl}^1 \in k$. Comparison with (3.3.4) gives

$G - i - 3j - 8k \geq 0$ for all combinations i, j, k corresponding to $l = 0$ in (3.3.4). Similarly $G - 3i - 5j - 4k \geq 0$ for i, j, k -combinations when $l = 0$ for contravariants.

Q.E.D. lemma 3.2.4.

Proof of lemma 3.2.5.

Let C and B be evenweighted co- and contravariants respectively.

$$\text{Put } C_{\text{can}} = \sum_{ijkl} q_{ijkl} m^l \cdot U_{\text{can}}^i H_{\text{can}}^j \otimes_{\text{can}}^k \quad (3.3.4)$$

Lemma 3.2.4 gives that

$$L \stackrel{\text{def}}{=} C = \sum_{\substack{ij,k \\ l=0}} \tau \frac{g-i-3j-3k}{6} U^i H^j \otimes^k \cdot q_{ijk0}$$

is a well-defined covariant which has m as factor on canonical form (where combinations of i, j, k, l correspond to those of (3.3.4)).

Lemma 3.2.2, a, b, and c gives that $(m-1)$, $(m-\epsilon)$, $(m-\epsilon^2)$ also are factors of L_{can} , where $\epsilon = e^{i2\pi/3}$. This means that $S_{\text{can}} = m(m-1)(m-\epsilon)(\epsilon^2-m) = m-m^4$ is factor in L_{can} . Since S has no multiple factor, and no factors in common with the discriminant, S divides L in general.

Analogous usage of lemma 3.2.2.d gives the result for the contravariant B .

Proof of proposition 3.2.6.

Lemma 3.2.5 gives that an even-weighted covariant C can be written as

$$C = \sum_{ijk} q_{ijk} \tau \frac{G-i-3j-8k}{6} U^i H^j \otimes^k + S \cdot C_1,$$

where C_1 is either 0, or a covariant of bidegree $(G-4, 0)$.

If C_1 is a constant, are we through; if not, we have weight C_1 is $G-4-0/3 = \text{weight } C-4$ is even, and we repeat the process

on C_1 . After maximally $\lceil \frac{G}{4} \rceil$ steps, the process stops, and we have reached the conclusion of the lemma for covariants.

The "contravariant case" is treated in the same way.

Proof of lemma 3.2.7.

Let C and B be odd-weighted co- and contravariants respectively.

Independent of the weight-assumption we obtained in lemma 3.2.1 that

$$C_{\text{can}} = P_0(m, x^3, y^3, z^3) + xyzP_1(m, x^3, y^3, z^3) + (xyz)^2P_2(m, x^3, y^3, z^3).$$

Similar as in lemma 3.2.2 we now obtain that P_0, P_1, P_2 are anti-symmetric in their 3 last indeterminates. This means that

$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3)$ is factor in C_{can} . Since $J_{\text{can}} = (1 + 8m^3)^3(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$, this means that J_{can} is factor in $(DC)_{\text{can}}$. ($D_{\text{can}} = (1 + 8m^3)^3$).

In order to generalize to general form, we must show that J has no multiple factors, and none common with D on general form. This is not immediate since the statement is not valid on canonical form. If J had multiple factors on general form, this would be the case on canonical form; and therefore we can conclude that no such eventual factor can contain x , y , or z , since there are 9 different ones of these on canonical form. Therefore we are in a position to use lemma 3.1.1 if we can show that J has no factors only including $a, b, c, a_2, \dots, c_{2,m}$.

The plane cubics constitute a \mathbb{P}_k^9 . It is enough to show that the curves on which J vanishes identically, constitutes a subset of codimension ≥ 2 .

Since $J_{\text{can}} = (1+8m^3)^3(x^3-z^3)(x^3-y^3)(z^3-y^3)$, J_{can} does not vanish identically on the non-singular curves; $(1+8m^3)^3 \neq 0$, when a curve is non-singular.

From the canonical form, $J(1,1,1,0,\dots,0,m,x,y,z)$, we can construct $J(a,b,c,0,\dots,0,m,x,y,z)$, using lemma 3.1.3. This gives:

$$J(a,b,c,0,0,\dots,0,m,x,y,z) = (abc+8m^3)^3(cz^3-ax^3)(by^3-ax^3)(cz^3-by^3).$$

We here remark that the reasoning for contravariants goes on in an analogous way all the way through the proof of this lemma, with α, β, γ in the place of x, y, z , and K in the place of J . For $K(a,b,c,0,\dots,0,m,\alpha,\beta,\gamma)$, we obtain $(abc+8m^3)^3(\alpha\gamma^3-c\alpha^3)(a\beta^3-b\alpha^3)(b\gamma^3-c\beta^3)$. We put $b = c = m = 1$, $a = 0$. Then "the original curve", U , becomes: $y^3 + z^3 + 6xyz = 0$. This is a curve with a node in $(1,0,0)$, and neither J nor K vanishes identically for this node-curve. This means that J (or K) only vanishes on a set of curves included in cusp-curves and reducible curves. Cusp-curves correspond to $T = S = 0$, and is a set of curves of codimension 2. Reducible curves constitute a $\mathbb{P}_k^5 \times \mathbb{P}_k^2$ which is also of codimension 2.

Therefore neither J nor K possesses factors only containing a, b, c, \dots, m . Therefore J divides DC (K divides DB) in general when $C(B)$ is odd-weighted. But since $J(K)$ has no factors in common with D , J divides C (K divides B).

Q.E.D.

We have now obtained the results of theorems 2.2.1 and 2.2.2, with one important exception; the statements about relations.

CHAPTER 4

PROOF OF THEOREMS (PART 2).

We now will find the relations between the generators we found in chapter 3.

Proposition 4.1.

a. S, T, U, H, Θ are alg.indep/ k .

b.) S, T, P, Q, F are alg.indep/ k .

Proof.

a.) $\text{Tr.d } k[S, T, U, H, \Theta] : k[S, T] = 3$ which is true on canonical form, and therefore in general (see lemma 3.1.2).

Therefore it is enough to show: $\text{Tr.d } k[S, T] : K = 2$. Assume $\exists G \in k[X, Y]$, s.t. $g(S, T) = 0$. We can assume G does not have X or Y as factor because $S^i T^j$ contains $(abc)^2 j (abc m)^i$, which is not 0, as an additive term. Hence there are some terms in G , containing only X or some containing only Y (or none of them). If some with only Y exist, $S = 0 \Rightarrow$

$$a_n T^n + a_{n-1} T^{n-1} + \dots + a_s = 0, \quad a_n, \dots, a_s \in k, \text{ which means}$$

that only a finite number of T -values is possible when $S = 0$.

$$a = b = c = r, \quad a_2 = a_3 = \dots = c_2 = m = 0 \text{ gives } S = 0,$$

$T = r^6$, r arbitrary, so this is a contradiction.

We obtain a similar contradiction assuming that some terms with only X exist, so there can be no such G . This proves part a.)

b.) Similar.

Proposition 4.2.

The relations

$$J^2 = 4\Theta^3 + TU^2\Theta^2 + \Theta(-4S^3U^4 + 2STU^3H - 72S^2U^2H^4 - 18TUH^3 + 108SH^4) - 16S^4U^5H - \\ - 11S^2TU^4H^2 - 4T^2U^3H^3 + 54STU^2H^4 - 432S^2UH^5 - 27TH^6$$

and

(4.2.1)

$$K^2 = \frac{1}{16}(T^2 + 64S^3)F^3 - \frac{1}{8}TF^2Q^2 - 12S^2F^2PQ + \frac{9}{2}STF^2P^2 + \frac{1}{16}FQ^4 + \frac{15}{2}SFQ^2P^2 - \\ - \frac{9}{2}TFQP^3 - 27S^2FP^4 + \frac{1}{2}Q^3P^3 + 54SQP^5 - 27TP^6$$

are valid, and are the only polynomial relations between S, T, U, H, Θ, J and S, T, P, Q, F, K , respectively.

Proof of prop. 4.2.

It is clear that J^2 and K^2 can be uniquely expressed in S, T, U, H, Θ and S, T, P, Q, F since they are even-weighted, and since each of the two sets are alg.indep./k.

In [1], Salmon has calculated the first relation, and I have copied his way of calculation, in order to express K^2 in S, T, P, Q, F .

I will now show that each polynomial relation between S, T, U, H, Θ, J contains (4.2.1) as a factor. The "contravariant case" can be treated analogously.

We define $\mathcal{K} \stackrel{\text{def}}{=} k(S, T, U, H, \Theta)$. J satisfies the polynomial relation

$$X^2 - [4\Theta^3 + T^2U^2\Theta^2 + \dots - 27TH^6] = 0 \quad (4.2.2)$$

To show that (4.2.2) is irreducible in $\mathcal{K}[X]$, it appears to be enough to show that

$R \stackrel{\text{def}}{=} 4\theta^3 + T^2U^2\theta^2 + \dots - 27TH^6$ is not a square in $k(S,T,U,H,\theta)$.

It cannot be so, because the largest degree in θ is three, which is an odd number.

It is now easy to verify that all polynomial relations between S,T,U,H,θ,J must contain (4.2.1) as a factor; $X^2 - R$ being the irreducible polynomial of J over \mathcal{K} .

For contravariants the proof is built on the fact that

$\frac{1}{16}(T^2 + 64S^3)F^3 - 1/8 TF^2Q^2 + \dots - 27TP^6$ cannot be a square in $k(S,T,P,Q,F)$, the largest degree in F being 3.

This gives the proposition.

CHAPTER 5

REMARKS ON THE j -INVARIANT

R. Hartshorne mentions in [3] an entity called the j -invariant (which has nothing with the mentioned J -covariant to do).

After a change of coordinates a plane non-singular cubic can be written

$$y^2z = x(x-z)(x-\lambda z), \quad \lambda \in k,$$

i.e. a different canonical form from that Salmon (and I) has used. The j -invariant is defined as

$$j(\lambda) = \frac{2^8(\lambda^2 - \lambda + 1)}{\lambda^2(\lambda - 1)^2}.$$

According to Hartshorne, this j classifies a non-singular cubic up to projective equivalence (Hartshorne has a more general view-

point, classifying elliptic curves). There are altogether 6 different λ 's (in general) that give the same j -value, corresponding to 6 projectively equivalent canonical curves. Further on, there are as many equivalence classes as elements of k .

I will now define the j -invariant for a (not necessarily canonical) non-singular cubic in general.

If we enlarge our concept of invariants a bit, we can talk about invariants that are not necessarily polynomials, but rational function in a, \dots, m . These invariant must satisfy:

$$I(g \cdot a, \dots, g \cdot m) = I(a, \dots, m) \quad (5.11)$$

for all $g \in SL_3$, or all $g \in GL_3$, depending on how "strong" the invariance is expected to be. I now go for an invariant in this sence, satisfying (5.11) for all $g \in GL_3$, coinciding with Harts-horne's for his canonical curves. Calculations, testing, and failing gives that

$$j(a, \dots, m) = \frac{2^{10} \cdot 11 \cdot 13 \cdot S^3}{T^2 + 64S^3} \text{ coincides with}$$

the j -invariant. We take this as a definition.

Proposition 5.1.

j classifies all non-singular plane cubics. There are as many classes as there are elements of k .

Proof of prop. 5.1.

That j takes equivalent curves to equal values, is clear, since j is the quotient of two invariants, the weights of the nominator and the denominator being equal.

To show that non-equivalent curves get different j -values, it is enough to regard canonical curves;

$$x^3 + y^3 + z^3 + 6mxyz = 0.$$

Which curves correspond to a given j -value, j_0 ? (We disregard the irrelevant factor $2^{10} \cdot 11 \cdot 13$)

$$\frac{[S(m)]^3}{T^2(m) + 64S(m)^3} = j_0. \quad \text{This is equivalent to:} \quad (5.1.2)$$

$$m^{12} + (512j_0 - 3)m^9 + (192j_0 + 3)m^6 + (24j_0 - 1)m^3 + j_0 = 0.$$

I.e.: Maximally 12 different m 's give the same j .

We also observe that to each value of j , there is at least one m .

Lemma 3.1.3 and 3.2.1 gives that to a curve with m -value m_0 , there are projectively equivalent curves with m -values:

ϵm_0	$\frac{1 - m_0}{1 + 2m_0}$	$\frac{1 - \epsilon m_0}{1 + 2\epsilon m_0}$	$\frac{1 - \epsilon^2 m_0}{1 + 2\epsilon^2 m_0}$	
$\epsilon^2 m_0$	$\frac{\epsilon(1 - m_0)}{1 + 2m_0}$	$\frac{\epsilon(1 - \epsilon m_0)}{1 + 2\epsilon m_0}$	$\frac{\epsilon(1 - \epsilon^2 m_0)}{1 + 2\epsilon^2 m_0}$	$\epsilon = e^{i \cdot 2\pi/3}$
	$\frac{\epsilon^2(1 - m_0)}{1 + 2m_0}$	$\frac{\epsilon^2(1 - \epsilon m_0)}{1 + 2\epsilon m_0}$	$\frac{\epsilon^2(1 - \epsilon^2 m_0)}{1 + 2\epsilon^2 m_0}$	

s.t. (maximally) 11 others are equivalent to the original one.

Pick a j -value j_0 , and a m -value m_0 , that gives the j -value j_0 . We make the equation in m , that m should be

either m_0 , or one of the 11 that we just listed:

$$(m-m_0)(m-\epsilon m_0) \dots (m - \epsilon^2 \frac{1-\epsilon^2 m_0}{1+2\epsilon^2 m_0}) = 0. \quad (5.1.3)$$

A long calculation gives that the equations (5.1.2) and (5.1.3) are the same. This proves the proposition.

This also gives an interpretation of j as the product of the different m 's corresponding to a given equivalence class (we disregard the factor $2^{10} \cdot 11 \cdot 13$ once again).

References:

- [1] G. Salmon: Higher Plane Curves
- [2] G. Salmon: Modern Higher Algebra
- [3] R. Hartshorne: Algebraic Geometry
- [4] Journal für die reine und angewandte Mathematik (1855).

Appendix on binary quartics.

All the work with ternary cubics was based on the fact that such a curve could be written on a canonical form with only one coefficient, m , after an action of a suitable element of GL_3 (in the generic, nonsingular case).

One could ask in what other cases the method of a canonical form with only one coefficient, could be used. If n is the number of variables, x, y, z, \dots , and r is the degree of the hypersurface in \mathbb{P}^{n-1} , one finds that this is only possible when

$$\binom{n+r-1}{r} - n^2 = 1.$$

One checks immediately that $n = 3, r = 3$, and $n = 2, r = 4$, are the only cases. This indicates the possibility of computing the covariants and contravariants of binary quartics by the same methods as for ternary cubics. In fact, it can be done, and this is no new result. For binary polynomials, we have other and more elegant ways to compute the covariant ring for low r , but just for fun, we will prove the following theorem, which is given, f.ex. in Springer's lecture notes, nr. 585, p. 61 (T.A. Springer).

Theorem 1.

For binary quartics the covariant ring is given as $k[S, T, U, H, J]$, where the bidegrees are $(3, 0), (2, 0), (1, 4), (2, 4), (3, 6)$. The only relation between these is $J^2 = 4H^3 - SU^3 - TU^2H$.

U is "the polynomial itself". H is the Hessian of U (up to scalars) $S = 0$ expresses that U is a sum of two fourth powers.

$T = \tau_4(U, U)$, $J = \tau_1(U, H)$. (Transvectants, see Springer).

H, S, T, J are defined up to scalar factors.

Sketch of proof: Any quartic without a repeated root can be written on the canonical form

$$U_{\text{can}} = x^4 + y^4 + 6mx^2y^2. \quad \text{We get}$$

$$S_{\text{can}} = m - m^3, \quad T_{\text{can}} = 1 + 3m^2, \quad H_{\text{can}} = m(x^4 + y^4) + (1 - 3m^2)x^2y^2.$$

$$J_{\text{can}} = (1 - 9m^2)xy(x^4 - y^4). \quad \text{The lemma sequence becomes:}$$

Analogue of lemma 3.2.1.

Any covariant can on canonical form be written as

$$xyP_1(x^4, y^4) + (xy)^3P_3(x^4, y^4) \quad \text{if weight } C \text{ is odd}$$

$$P_0(x^4, y^4) + (xy)^2P_2(x^4, y^4) \quad \text{if weight } C \text{ is even.}$$

P_i are polynomials for $i = 0, 1, 2, 3$.

$$\text{One uses } g_1 = \begin{pmatrix} 1 & 0 \\ 0 & i^3 \end{pmatrix}, \quad g_2 = \begin{bmatrix} i^3 & 0 \\ 0 & 1 \end{bmatrix} \quad i = e^{i\pi/2}.$$

Analogue of lemma 3.2.2.

Even-weighted covariants may be written

$$C_{\text{can}} = \sum_{ijl} m^l (x^4 + y^4)^i (x^2 y^2)^j \cdot q_{ijl}. \quad (\text{I.1.})$$

$$(\text{one uses } g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$$

Analogue of lemma 3.2.3 (weight C is even).

$$\text{Any covariant can be written: } C_{\text{can}} = \sum m^l U^i H^j q_{ijl} \quad (\text{I.2.})$$

One uses $g = \begin{pmatrix} 1 & i \\ i^3 & i^2 \end{pmatrix}^{-1}$ (analogously to lemma 3.1.3)

to handle the factor $1 - 3m$ of the discriminant, $D_{can} = 1 - 9m^2$, and a similar g to handle $1 + 3m$. The analogue of the formula $(G-1) - (i+2k) \geq 0$ is $(G-1) - i \geq 0$, now referring to (i.1.)

Analogue of lemma 3.2.4.

If an even-weighted covariant is written on the form (I.2.), for $l = 0$, the corresponding (i,j) -combinations give

$$G - i - 2j \text{ is even, and } G - i - 2j \geq 0.$$

Proof: Similar to that of 3.2.4.

Analogue of 3.2.5.

Referring to (I.2).

$$L_{can} \stackrel{\text{def}}{=} (C - \sum_{l=0}^m q_{ijl} \cdot T^{\frac{G-i-2j}{2}} U^i H^j)_{can} \text{ has } S_{can} \text{ as factor.}$$

Proof: The polynomials $x^4 + y^4$, $x^4 + y^4 + 6 \cdot 1 x^2 y^2$, $x^4 + y^4 + 6 \cdot (-1) x^2 y^2$ are projectively equivalent, so:

$$L_{can}(0, x, y) = 0 \Rightarrow L_{can}(1, x, y) = 0 \Rightarrow L_{can}(-1, x, y) = 0$$

$$\Rightarrow -m(m-1)(m+1) = S_{can} \text{ is factor in } L_{can}.$$

One generalizes to general form as for ternary cubics.

Analogue to lemma 3.2.7.

An odd weighted covariant has J as a factor.

Proof: xy is a factor, and $x^4 - y^4$ is a factor in C_{can} .

The rest of the proof is similar to that in 3.2.7..

The relation $J^2 = 4H^3 - SU^3 - TV^2H$ is computed on canonical form, and then generalized to general form.

Making the remaining analogues is left to the reader, knowing that:

$$S = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3$$

$$T = a_0 a_4 - 4a_1 a_3 + 3a_2^2$$

when

$$U = a_4 x^4 + 4a_3 x^3 y + 6a_2 x^2 y^2 + 4a_1 x y^3 + a_0 y^4.$$